

A PARTITIONING AND RELATED PROPERTIES FOR THE QUOTIENT COMPLEX $\Delta(B_{lm})/S_l \wr S_m$

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ABSTRACT. We study the quotient complex $\Delta(B_{lm})/S_l \wr S_m$ as a means of deducing facts about the ring $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$. It is shown in [He] that $\Delta(B_{lm})/S_l \wr S_m$ is shellable when $l = 2$, implying Cohen-Macaulayness of $k[x_1, \dots, x_{2m}]^{S_2 \wr S_m}$ for any field k . We now confirm for all pairs (l, m) with $l > 2$ and $m > 1$ that $\Delta(B_{lm})/S_l \wr S_m$ is not Cohen-Macaulay over $\mathbb{Z}/2\mathbb{Z}$, but it is Cohen-Macaulay over fields of characteristic $p > m$ (independent of l). This yields corresponding characteristic-dependent results for $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$. We also prove that $\Delta(B_{lm})/S_l \wr S_m$ and the links of many of its faces are collapsible, and we give a partitioning for $\Delta(B_{lm})/S_l \wr S_m$.

1. INTRODUCTION

Let B_n denote the Boolean algebra of subsets of $\{1, \dots, n\}$ ordered by inclusion. The natural symmetric group action on $\{1, \dots, n\}$ induces a rank-preserving, order-preserving action on B_n . Likewise, the wreath product of symmetric groups $S_l \wr S_m \subset S_{lm}$ acts on the Boolean algebra B_{lm} . (Recall that $S_l \wr S_m$ is the subgroup of S_{lm} of order $(l!)^m m!$ which permutes the values $il + 1, \dots, (i+1)l$ among themselves for each $0 \leq i < m$ and also wholesale permutes these m sets of size l .) This induces an $S_l \wr S_m$ -action on chains $\hat{0} < u_0 \cdots < u_i < \hat{1}$ of comparable poset elements, i.e. on faces in the order complex $\Delta(B_{lm})$. The action on chains gives rise to a quotient cell complex, denoted $\Delta(B_{lm})/S_l \wr S_m$, which consists of the $S_l \wr S_m$ -orbits of the order complex faces. As a word of caution, the quotient complex $\Delta(B_{lm})/S_l \wr S_m$ does not coincide with the order complex of the quotient poset $B_{lm}/S_l \wr S_m$ (cf. [BK] for a study of which quotient complexes are order complexes of quotient complexes), because there are covering relations $u \leq v, u' \leq v'$ in B_{lm} belonging to distinct orbits despite having $u' = gu$ and $v' = g'v$ for some $g, g' \in S_l \wr S_m$.

We will rely on results of Stanley, Hochster-Eagon, Reiner, Björner and Garsia-Stanton to transfer properties of the quotient complex $\Delta(B_{lm})/S_l \wr S_m$ into algebraic facts about the subring of invariant polynomials $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$. Section 2 will review these results about subrings of invariant polynomials, quotient complexes and more generally about simplicial posets from [Bj], [GS], [HE], [Re] and [St3]. Sections 3 and 4 follow up on previous work in [He], where a lexicographic shelling was given for $\Delta(B_{2m})/S_2 \wr S_m$. In section 3, we show that $\Delta(B_{lm})/S_l \wr S_m$ is not Cohen-Macaulay over the integers mod 2 whenever $l > 2$ and $m > 1$, by exhibiting local 2-torsion. (The situation is trivial whenever $l = 1$ or $m = 1$.) Section 4 shows that $\Delta(B_{lm})/S_l \wr S_m$ and many of its links are collapsible, and finally we provide a partitioning for $\Delta(B_{lm})/S_l \wr S_m$ in Section 5.

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One theme that runs throughout this paper is the use of ideas typically associated (at least implicitly) to lexicographic shellings to deduce properties related to shellability for complexes that are not shellable; in particular, we give collapsibility, Cohen-Macaulayness (for certain field characteristics) and partitionability results for $\Delta(B_{lm})/S_l \wr S_m$. In theory, our partitioning for $\Delta(B_{lm})/S_l \wr S_m$ gives a Hilbert series expression for $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$, but it would be desirable to find a simpler expression. Our partitioning for $\Delta(B_{lm})/S_l \wr S_m$ is very similar to the latter half of the (very complicated) partitioning argument used in [He] for $\Delta(\Pi_n)/S_n$; one of our goals was to simplify that argument.

It remains open for $l > 2$ to determine for which field characteristics p such that $2 < p \leq m$ the ring $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ is Cohen-Macaulay. Garsia-Stanton showed in [GS] how to deduce Cohen-Macaulayness over fields of characteristic p from partitionings in which p does not divide the determinant of the incidence matrix. We hope that our work may help with the resolution of this question.

2. SIMPLICIAL POSETS, QUOTIENT COMPLEXES AND SUBRINGS OF INVARIANT POLYNOMIALS

Boolean cell complexes were defined as follows in [Bj] and [GS]:

Definition 2.1. *A regular cell complex is **boolean** if every lower-interval in its face poset is a Boolean algebra, namely if each cell has the combinatorial type of a simplex.*

Stanley studied their face posets, which he called simplicial posets, in [St3]. People often use the term simplicial poset to mean either the face poset or the cell complex itself; we will reserve the term simplicial poset exclusively for the face posets, to emphasize the distinction between a boolean cell complex and the order complex of its face poset.

One may think of the cells in a boolean cell complex as simplices, but unlike in simplicial complexes, multiple faces may have the same set of vertices. As a result, two faces may overlap in a simplicial complex rather than simply in a face. We refer to i -cells as i -faces, 0-cells as vertices, and call cells of top dimension facets. Our interest is in a particular class of boolean cell complexes, namely the quotient complexes Δ/G made up of the G -orbits of faces in a simplicial complex Δ when a group G acts simplicially on the faces of Δ .

Stanley defined the face ring $k[P]$ for a simplicial poset P in [St3] by taking the faces in a boolean cell complex (or equivalently the elements in its face poset P) as the generators of a polynomial ring over a field k and giving the generators the following three types of relations:

- (1) xy if there is no face containing both x and y
- (2) $xy - (x \wedge y) \left(\sum_{z \in \text{lub}(x,y)} z \right)$, where $\text{lub}(x, y)$ denotes the set of least upper bounds of x and y
- (3) $\hat{0} - 1$

Stanley proved the following in [St3], using facts about algebras with straightening laws.

Theorem 2.1 (Stanley). *The face ring $k[P]$ of a Cohen-Macaulay simplicial poset P is a Cohen-Macaulay ring.*

Let us denote the face ring of the face poset of a quotient complex Δ/G by $k[\Delta/G]$. In [Re], Reiner established the following connection between face rings of quotient complexes and subrings of invariant polynomials (cf. [St4, p. 53] for the definition of $k[\Delta]$, or specialize the above definition to simplicial complexes).

Theorem 2.2 (Reiner). *The rings $k[\Delta/G]$ and $k[\Delta]^G$ are isomorphic.*

Reiner also showed (unpublished) that Cohen-Macaulayness for subrings of invariant polynomials for face rings of certain quotients of type A Coxeter complexes transfers to Cohen-Macaulayness of other subrings of invariant polynomials. A proof of the following result has been provided by Reiner in an appendix.

Theorem 2.3 (Reiner). *If $G \subset S_n$ and $k[\Delta(B_n)]^G$ is Cohen-Macaulay over a field k , then $k[x_1, \dots, x_n]^G$ is Cohen-Macaulay over the same field k .*

In [Bj4], Björner established a notion of shellability for boolean cell complexes (stated slightly differently than below) and noted that it implies Cohen-Macaulayness.

Definition 2.2 (Björner). *A pure boolean cell complex is **shellable** if the facets may be ordered F_1, \dots, F_k so that $F_j \cap (\cup_{i=1}^j F_i)$ is pure of codimension one for each $1 < j \leq k$.*

Just as in the case of simplicial complexes, this is equivalent to requiring there to be a unique minimal new face at each facet insertion.

Proposition 2.1 (Björner). *If a pure boolean cell complex is shellable, then the underlying topological space is Cohen-Macaulay (over any field).*

We will also use the following result of Hochster and Eagon to get at the Cohen-Macaulayness of $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ for relatively large field characteristics.

Theorem 2.4 (Hochster-Eagon). *If Δ is a Cohen-Macaulay simplicial complex and the characteristic of k does not divide $|G|$, then $k[\Delta]^G$ is a Cohen-Macaulay ring.*

In discussing which complexes $\Delta(B_{lm})/S_l \wr S_m$ are shellable, we will make use of the fact that $\Delta(B_{lm})/S_l \wr S_m$ is balanced. Recall that a boolean cell complex of dimension $d - 1$ is **balanced** if there is a map $\kappa : V(\Delta) \rightarrow \{1, \dots, d\}$ that colors the vertices with d colors so that no two vertices in the same face are the same color. We refer to the set of colors for the vertices in a face as the **support** of the face. Notice that the order complex of a finite, graded poset is balanced by poset rank. One nice feature of balanced complexes is that their face rings have very explicit linear systems of parameters (l.s.o.p.'s), namely the face ring of a balanced $(d - 1)$ -dimensional complex Δ has linear system of parameters $\theta_1, \dots, \theta_d$ in which $\theta_i = \sum_{v: \kappa(v)=i} v$ (cf. [St4]).

If a complex Δ of dimension $d - 1$ is shellable and $k[\Delta]$ has linear system of parameters $\theta_1, \dots, \theta_d$, then $k[\Delta] = \coprod_{\nu \in X} \nu \cdot k[\theta_1, \dots, \theta_d]$ and the set X of minimal faces in the shelling is a k -basis for $k[\Delta]/(\theta_1, \dots, \theta_d)$ (cf. [St4]). In this case, X is called a **basic set** for $k[\Delta]$. Garsia and Stanton use shellings and certain types of partitionings as a means for constructing basic sets for rings $k[\Delta/G]$ and for related subrings of invariant polynomials in [GS]; we follow their notation in the remainder of this section.

If c is a face of Δ consisting of vertices x_{i_1}, \dots, x_{i_r} , then denote by $x(c)$ the monomial $x_{i_1} \cdots x_{i_r}$ in the face ring $k[\Delta]$. When a group H acts on Δ , the Reynold's operator R^H acts on $k[\Delta]$ by

$$R^H(x(c)) = \frac{1}{|H|} \sum_{h \in H} hx(c) = \frac{1}{|H|} \sum_{h \in H} x(hc).$$

A set of chain monomials $\{x(b)|b \in B\}$ given by a collection B of chains in a poset P is called a **basic set** if every element Q of the Stanley-Reisner ring $k[\Delta]$ has a unique expression

$$Q = \sum_{b \in B} x(b)Q_b(\theta_1, \dots, \theta_d)$$

where the coefficients $Q_b(\theta_1, \dots, \theta_d)$ are polynomials with rational coefficients in the variables $\theta_1, \dots, \theta_d$. This yields a Hilbert series expression

$$\text{Hilb}(k[\Delta], \lambda) = \left(\prod_{i=1}^d \frac{1}{1 - \lambda^{\deg(\theta_i)}} \right) \left(\sum_{b \in B} \lambda^{\deg(x(b))} \right).$$

All Cohen-Macaulay posets have such basic sets.

Theorem 2.5 (Garsia-Stanton). *If Δ/H has a shelling F_1, \dots, F_k where G_j is the unique minimal new face in $F_j \setminus (\cup_{i < j} F_i)$ and b_j is a representative of the orbit G_j within Δ , then the orbit polynomials $R^H x(b_i)$ form a basic set for the subring of invariant polynomials $k[\Delta]^H$, implying Cohen-Macaulayness over any field.*

When a subgroup G of the symmetric group S_n acts on the boolean algebra B_n in a rank-preserving, order-preserving fashion, then Garsia and Stanton proved in [GS] that basic sets for $k[\Delta(B_n)]^G$ transfer to basic sets for $k[x_1, \dots, x_n]^G$ and that certain types of partitionings (including all shellings) give rise to basic sets. We state their result in Theorem 2.6, but first we give a definition it will use.

Definition 2.3. *The **incidence matrix** of a partitioning is a matrix with rows indexed by facets and columns indexed by the minimal faces in the partitioning. If $G_j \subseteq F_i$ then $A_{i,j} = 1$ and otherwise $A_{i,j} = 0$.*

The incidence matrix for a partitioning coming from a shelling is always upper triangular with 1's on the diagonal, hence nonsingular (over any field). Other partitionings may yield incidence matrices that are singular over finite fields of sufficiently small characteristic. It is possible to construct partitionings with singular incidence matrices for Cohen-Macaulay complexes (personal communication of Reiner), so one cannot conclude non-Cohen-Macaulayness by obtaining a singular incidence matrix.

Theorem 2.6 (Garsia-Stanton). *Let $G \subset S_n$ act as above and let $[G_1, F_1] \cup \dots \cup [G_k, F_k]$ be a partitioning for $\Delta(B_n)/G$ with nonsingular incidence matrix. Then $x(G_1), \dots, x(G_k)$ form a basic set for $k[\Delta(B_n)/G]$, w.r.t. the l.s.o.p. $\theta_1, \dots, \theta_{n-1}$ given by the balancing. Sending θ_i to the elementary symmetric function e_i and $G_j = S_1 \subset S_2 \subset \dots \subset S_r$ to the product $x_{S_1} x_{S_2} \cdots x_{S_r}$, in which $x_S = \prod_{i \in S} x_i$ yields a basic set for $k[x_1, \dots, x_n]^G$.*

3. SHELLABILITY AND COHEN-MACAULAYNESS RESULTS

Using a lexicographic shellability criterion for pure, balanced complexes, it is shown in [He] that $\Delta(B_{2n})/S_2 \wr S_n$ is shellable. Below we will describe the lexicographic order that led to this shelling, but we refer readers to [He] for the proof that it does indeed give a shelling.

The following chain-labeling for $\Delta(B_{2m})/S_2 \wr S_m$ gives a lexicographic shelling: label the covering relation $\{\sigma_1, \dots, \sigma_{i-1}\} \prec \{\sigma_1, \dots, \sigma_{i-1}, \sigma_i\}$ in the poset B_{2m} with the label $\sigma_i \in \{1, \dots, 2m\}$, recording the insertion of σ_i . Thus, the saturated chain $\emptyset \prec \{\sigma_1\} \prec \dots \prec \{\sigma_1, \dots, \sigma_{2m}\}$ is labeled $\sigma_1 \dots \sigma_{2m} \in S_{2m}$. The facets in $\Delta(B_{2m})/S_2 \wr S_m$ are the orbits of the saturated chains in B_{2m} , and by convention we label each of these orbits with lexicographically smallest permutation among the labels for members of the orbit. This chain-labeling gives a *CC*-shelling, in the sense developed for posets by Kozlov in [Ko1] and extended to pure, balanced complexes in [He]. (Hultman recently further generalized the lexicographic shellability criterion of [He] to non-pure balanced complexes in [Hu].)

The labels for the orbit representatives turn out to be the permutations of $1, \dots, 2m$ which do not have any inversion pairs $(2i-1, 2i)$ or $(2i-1, 2i+1)$, namely permutations in which the odd numbers appear in increasing order and each odd number comes earlier than its even successor.

Example 3.1. The orbit representatives for $\Delta(B_6)/S_2 \wr S_3$, listed in lexicographic order, are 123456, 1235•46, 123◦56•4, 13•2456, 13•25•46, 13•256•4, 1◦34•256, 1◦345•26, 1◦3456•2, 135•246, 13◦5•26•4, 1◦35•4•26, 1◦35•46•2, 13◦56•24, and 1◦3◦56•4•2. Hollow dots denote ascents which behave topologically like descents and filled-in dots indicate traditional descents. The minimal new face for a facet is the union of the ranks of the hollow dots and the ranks of the filled-in dots. For instance, the swap ascent in $1◦3456•2$ comes from a codimension one face skipping rank 1 in the intersection of 134562 with 132564, resulting from the fact that 312564 is in the same orbit as 134562.

To describe the group $S_2 \wr S_m$ (and more generally $S_l \wr S_m$), let us first place the numbers $1, \dots, 2m$ (resp. $1, \dots, lm$) in a $2 \times m$ (resp. $l \times m$) table, by sequentially inserting the numbers from left to right in each row, proceeding from one row to the next from top to bottom, as in Figure 1. The elements of $S_2 \wr S_m$ (resp. $S_l \wr S_m$) may then be described as the permutations in S_{2m} (resp. S_{lm}) which permute the numbers within each row and then permute the set of rows. More

1	2
3	4
5	6

FIGURE 1. Labeled boxes acted upon by $S_2 \wr S_3$

formally, each element of $S_2 \wr S_m$ is a composition of $\pi_2 \circ \pi_1 \in S_{2m}$ in which $\pi_1 = (12)^{e_1}(34)^{e_2} \dots (2m-1, 2m)^{e_m}$ for some vector $(e_1, \dots, e_m) \in \{0, 1\}^m$ and where π_2 is obtained from some $\pi \in S_m$ by requiring $\pi_2(2i) = 2\pi(i)$ and $\pi_2(2i-1) = 2\pi(i)-1$ for $1 \leq i \leq m$ (and of course there is a similar definition for $l > 2$).

The lexicographic shelling for $\Delta(B_{2m})/S_2 \wr S_m$ may be combined with results of Stanley and Reiner (recalled in Theorems 2.1 and 2.2, respectively) to obtain Cohen-Macaulayness for $k[x_1, \dots, x_{2m}]^{S_2 \wr S_m}$ independent of field characteristic (or equivalently over the integers), as is noted in [He]. When $\text{char}(k) = 0$, this is a special case of a result from [HE], but the shelling also allows coefficients in fields of finite characteristic or the integers. By Theorem 2.6, the lexicographic shelling for $\Delta(B_{2n})/S_2 \wr S_n$ also yields a basic set for the subring $k[x_1, \dots, x_{2m}]^{S_2 \wr S_m}$ of invariant polynomials. A simple description of which descent sets occur in the lexicographic shelling would be desirable in that it would yield a nice description of these basic sets (and in turn a nice Hilbert series expression).

The story is more subtle for $\Delta(B_{lm})/S_l \wr S_m$ when l is greater than 2. It is observed in [He] that these complexes cannot be shellable when $l > 2$, by a Molien series computation which shows that the Hilbert series disagrees with the expression that would result from applying Theorem 2.5 (recalled from [GS]) to any potential shelling order. Now we construct explicit faces whose links have 2-torsion and give partial results regarding the question of for which coefficient fields is $\Delta(B_{lm})/S_l \wr S_m$ Cohen-Macaulay. In particular, the fact that $\Delta(B_{lm})$ is a triangulation of a sphere immediately implies (via a result of Hochster and Eagon [HE]) that $\Delta(B_{lm})/S_l \wr S_m$ is Cohen-Macaulay for coefficient fields of characteristic p so long as p does not divide $|S_l \wr S_m|$, i.e. for primes p larger than $\max(l, m)$. We will do slightly better, showing Cohen-Macaulayness for $p > m$, regardless of how large l grows. We also show that $\Delta(B_{lm})/S_l \wr S_m$ and the links of many faces are collapsible, restricting how local p -torsion in lower homology might arise.

For each pair (l, m) with $l > 2$ and $m > 1$, we will provide a face F such that $\text{lk}(F)$ has dimension two and also has homology group $H_1(\Delta, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, precluding Cohen-Macaulayness. First consider the link of the face $F = \emptyset \subseteq \{1, 4\} \subseteq \{1, 2, 4, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$ in $\Delta(B_6)/S_3 \wr S_2$. Notice that $\text{lk}(F)$ has 3 vertices, 6 edges and 4 2-simplices, and that the underlying topological space is the real projective plane RP_2 .

Proposition 3.1. The quotient complex $\Delta(B_{lm})/S_l \wr S_m$ is not Cohen-Macaulay over $\mathbb{Z}/2\mathbb{Z}$ for $l > 2$ and $m > 1$, hence $\mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ is not Cohen-Macaulay for such pairs (l, m) .

PROOF. One gets RP_{l-1} as the link of a face in $\Delta(B_{2l})/S_l \wr S_2$, as follows: let us call all the letters in the first “row” 1 and all the letters in the second “row” 2 (since the letters in a row are all interchangeable) and then take the face

$$F = \emptyset \subseteq \{1, 2\} \subseteq \{1^2, 2^2\} \subseteq \dots \subseteq \{1^l, 2^l\}.$$

Note that the link of this face in $\Delta(B_{2l})/S_l \times S_2$ is a sphere, because $\Delta(B_{2l})/S_l \times S_2$ is lexicographically shellable (as shown by Garsia and Stanton in [GS]), and the restriction of this shelling to $\text{lk}(F)$ in $\Delta(B_{2l})/S_l \times S_2$ has one decreasing chain. We obtain the desired link in $\Delta(B_{2l})/S_l \wr S_2$ by gluing together pairs of antipodal faces in this sphere (i.e. by identifying faces in which the two classes of objects are exchanged); thus we obtain projective space in a completely natural fashion. This link also sits inside $\Delta(B_{lm})/S_l \wr S_m$ as the link of a larger face.

The conclusion about $\mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ for such pairs (l, m) follows from the same reasoning used for other coefficient fields in Proposition 3.2. \square

Proposition 3.2. *The quotient complex $\Delta(B_{lm})/S_l \wr S_m$ and consequently the ring $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ is Cohen-Macaulay over fields of characteristic p whenever $p > m$.*

PROOF. It is shown that $\Delta(B_{lm})/S_l \times \dots \times S_l$ is shellable (and hence Cohen-Macaulay over any field) in [GS]. Note that $\Delta(B_{lm})/S_l \wr S_m$ is the quotient of a Cohen-Macaulay complex by an S_m -action, by virtue of the isomorphism $\Delta(B_{lm})/S_l \wr S_m \cong (\Delta(B_{lm})/S_l \times \dots \times S_l)/S_m$. Thus, one may apply the result of Hochster and Eagon [HE], recalled in Theorem 2.4, to conclude that there is no p -torsion unless p divides $|S_m|$, i.e. unless $p \leq m$.

Stanley's result from [St3] that face rings of Cohen-Macaulay simplicial posets are Cohen-Macaulay then tells us that $k[\Delta(B_{lm})/S_l \wr S_m]$ is Cohen-Macaulay for $p = \text{char}(k) > m$, but $k[\Delta(B_{lm})/S_l \wr S_m] \cong k[\Delta(B_{lm})]^{S_l \wr S_m}$ by Theorem 2.2. Now we apply Theorem 2.3 to conclude that $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ is also Cohen-Macaulay for $p = \text{char}(k) > m$. \square

Question 3.1. *Is there any local p -torsion in lower homology for $2 < p \leq m$?*

In Section 5, we will give a partitioning for $\Delta(B_{lm})/S_l \wr S_m$, and if the determinant of the incidence matrix for this partitioning were not divisible by a prime p , then one could conclude Cohen-Macaulayness of $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ for $\text{char}(k) = p$. We suspect that p -torsion for primes larger than 2 would already appear in $\text{lk}(\emptyset < \{1, 2, 3\} < \{1^2, 2^2, 3^2\} < \{1^3, 2^3, 3^3\})$ if it ever occurs. The determinant of the incidence matrix for our partitioning of this link is $2^3 \cdot 3^5$, strongly suggesting (but not confirming) there is local 3-torsion present.

Remark 3.1. *The directed graph complexes studied by Kozlov in [Ko2] have faces whose links are isomorphic to $\text{lk}(\emptyset < \{1, \dots, m\} < \dots < \{1^1, \dots, m^1\})$ in $\Delta(B_{lm})/S_l \wr S_m$ for any pair (l, m) , and hence there is local 2-torsion arising just as in Proposition 3.1. Kozlov previously determined by computer that the directed graph complexes have local 2-torsion.*

4. COLLAPSIBILITY OF $\Delta(B_{lm})/S_l \wr S_m$ AND LINKS OF MANY FACES

This section proves that $\Delta(B_{lm})/S_l \wr S_m$ and the links of many of its faces are collapsible. The discussion of links is included in the hope that this might shed some light on the question of when the complexes are Cohen-Macaulay (i.e. for which field characteristics p such that $2 < p \leq m$). The collapsibility proofs are a relaxation of the sort of argument typically used to produce lexicographic shellings in that we will show that the intersection $F_j \cap (\cup_{i < j} F_i)$ of each facet F_j with the union of earlier ones is collapsible by exhibiting a topological ascent in each F_j , yielding a cone point in each intersection $F_j \cap (\cup_{i < j} F_i)$.

Theorem 4.1. *The quotient complex $\Delta(B_{lm})/S_l \wr S_m$ is collapsible.*

PROOF. Let us first order the saturated chains in B_{lm} lexicographically, just as in the lexicographic shelling for $\Delta(B_{2m})/S_2 \wr S_m$, and then choose the lexicographically earliest saturated chain in each $S_l \wr S_m$ -orbit as the orbit representative. Now we build up the quotient complex by sequentially inserting facets of $\Delta(B_{lm})/S_l \wr S_m$ in the resulting lexicographic order F_1, \dots, F_r . We will prove collapsibility by showing that each intersection $F_j \cap (\cup_{i < j} F_i)$ for $j > 1$ has a cone point so that collapsibility

is preserved with each facet insertion as we sequentially build the complex, since clearly F_1 is itself collapsible.

Let us encode the permutations in S_{lm} which label the saturated chain orbit representatives as words of content $\{1^l, 2^l, \dots, m^l\}$ by replacing the label $rl + s$ with the label $r + 1$ for each $0 \leq r < m, 0 < s < l$. Notice that this map is a bijection between permutations in S_{lm} which are lexicographically smallest in their $S_l \wr S_m$ -orbit and words of content $\{1^l, 2^l, \dots, m^l\}$ in which the first appearance of i precedes the first appearance of j for each pair $1 \leq i < j \leq m$. Note that any descent in the labels on a saturated chain orbit F_j may be replaced by a lexicographically smaller ascent to get the label for a lexicographically earlier saturated chain orbit F_i such that F_i and F_j share a codimension one face obtained by omitting the descent from F_j . We will show that $F_j \cap (\cup_{i < j} F_i)$ has a cone point at the element $u \in F_j$ just preceding the final appearance of m in the label for F_j .

First observe that the labels in F_j must be weakly increasing at u , since the latter label m is the largest value available. Suppose there is a maximal face $\sigma \in F_j \cap (\cup_{i < j} F_i)$ which omits u , and let $\sigma = \hat{0} = v_0 < v_1 < \dots < v_r < v_{r+1} = \hat{1}$. By the maximality of σ along with the fact that F_j is increasing at u , the labels on F_j must be weakly increasing from v_i to v_{i+1} for $0 \leq i \leq r$, since otherwise some interval has a descent which could be omitted from F_j to obtain a codimension one face $\tau \in F_j \cap (\cup_{i < j} F_i)$ such that $u \notin \tau$ and σ is strictly contained in τ (contradicting σ being maximal). Assume that σ is maximal in $F_j \cap (\cup_{i < j} F_i)$, σ omits u , and that F_j is weakly increasing between any two elements of σ . Let $F_{i'}$ be one of the facets that is lexicographically smaller than F_j and contains σ . For such an $F_{i'}$ to exist, we need there to exist a permutation π permuting the row values such that $F_j|_{\text{supp}(\sigma)} = \pi F_{i'}|_{\text{supp}(\sigma)}$ and such that $F_{i'}$ is lexicographically smaller than $\pi F_{i'}$. This guarantees us the following properties of σ :

- (1) σ must skip one or more intervals of F_j such that two of the labels $R_1, R_2 \in \{1, \dots, m\}$ each first appear in the first of these intervals. Let us assume $R_1 < R_2$.
- (2) Within each of the intervals of F_j skipped by σ the labels R_1 and R_2 appear an equal (nonzero) number of times
- (3) On each of the intervals skipped by σ , R_1 is the smallest label and R_2 is the largest label
- (4) The first appearance of R_2 labeling a covering relation $v \prec w$ such that $v, w \in \sigma$ is at a lower rank than the first such appearance of R_1
- (5) σ is missing at least one interval below v

Observe that a face σ meeting the above conditions cannot omit u because that would imply that $R_2 = m$, contradicting the fact that m must later appear as a label on the covering relation $v \prec w$, since we chose u to immediately precede the highest rank appearance of m as a label. \square

Remark 4.1. *This argument generalizes immediately to the link of any face which omits a single interval upon which the largest label appears more than once.*

We show next how to relax this requirement on the largest label to the requirement that some label appear more than once.

Proposition 4.1. *Let σ be a face that omits a single interval $T < S$ such that $S = T \cup S'$ and some letter in S' appears with multiplicity greater than one. Then $\text{lk}(\sigma)$ is collapsible.*

PROOF. Notice that any saturated chain F_j for $j > 1$ in $lk(\sigma)$ has a cone point in $F_j \cap (\cup_{i < j} F_i)$ located at the rank immediately before the last appearance of the largest label which does not appear exclusively in a rooted chain $T \prec u_1 \prec \cdots \prec u_r$ with strictly decreasing labels. For example, we claim that a saturated chain labeled 43212 has a cone point immediately before the second appearance of the label 2, since the labels 3 and 4 are eliminated by our requirement. The argument of the preceding theorem carries over easily to verify that this is a cone point, and the existence of such a rank follows from our requirements on σ . \square

Question 4.1. *Can collapsibility also be deduced for links which are not a single interval, when at least one (or perhaps all) the intervals in the link satisfy the conditions of Proposition 4.1? Notice that links of faces in quotient complexes are not simply joins of links of faces each omitting a single interval.*

5. PARTITIONING $\Delta(B_{lm})/S_l \wr S_m$

The complex $\Delta(B_{lm})/S_l \wr S_m$ is not shellable for $l > 2, m > 1$, but this section provides a partitioning for each pair l, m . This will involve a labeling that is quite a bit different from the one appearing in earlier sections.

Definition 5.1. *A partitioning of a pure boolean cell complex Δ is an assignment of a face G_i to each facet F_i so that the boolean upper intervals $[G_i, F_i]$ partition the set of faces in Δ , i.e. so that Δ is a disjoint union of boolean algebras $[G_1, F_1] \cup \cdots \cup [G_s, F_s]$ whose maximal elements are the facets of Δ .*

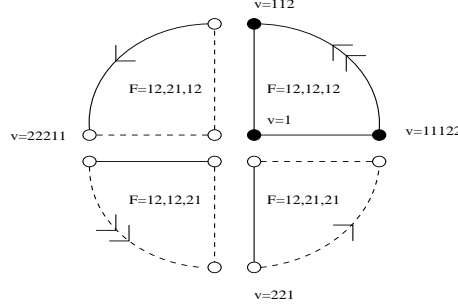
A partitioning of a pure, balanced complex Δ gives a combinatorial interpretation for the flag h -vector, namely each coordinate $h_S(\Delta)$ counts minimal faces G_i of support S in the partitioning. We begin with an example of how to partition a certain link which is not Cohen-Macaulay over the integers, before turning our attention to the entire complex $\Delta(B_{lm})/S_l \wr S_m$. Throughout this section, we use the isomorphism

$$\Delta(B_{lm})/S_l \wr S_m \cong (\Delta(B_{lm})/S_l \times \cdots \times S_l)/S_m$$

which allows us to view vertices as subsets of $\{1^l, \dots, m^l\}$ modulo an S_m -action permuting values. We refer to each of the m values as a **row**, motivated by the description of $S_l \wr S_m$ following Example 3.1.

Example 5.1. Consider the quotient complex $\Delta(B_6)/S_3 \wr S_2$ and the face $F = \emptyset < \{1, 2\} < \{1^2, 2^2\} < \{1^3, 2^3\}$. Notice that $lkF \cong RP_2$, as depicted in Figure 2 with the usual boundary identifications. Here, we represent the four facets by 3-tuples $(\sigma_1, \sigma_2, \sigma_3)$ of permutations in S_2 , written in one-line notation, with the requirement that $\sigma_1 = 12$. Vertices in the link are subsets S of $\{1^3, 2^3\}$ with the multiplicity of 1 and 2 differing by one, with the identification $S = (12)S$. We assign minimal faces to facets as follows: $(12, 12, 12)$ is assigned the empty chain, $(12, 12, 21)$ is assigned the chain $\{1\} < \{1^2, 2^3\}$, $(12, 21, 12)$ is assigned the chain $\{1, 2^2\} < \{1^3, 2^2\}$ and $(12, 21, 21)$ is assigned the chain $\{1\} < \{1, 2^2\}$. In Figure 2, vertices and edges of a facet that are assigned by the partitioning to a different facet are depicted by hollow circles and dashed edges, respectively.

The partitioning in the above example generalizes to $lk(\emptyset < \{1, \dots, m\} < \cdots < \{1^l, 2^l, \dots, m^l\})$ in $\Delta(B_{lm})/S_l \wr S_m$ by representing facets by l -tuples $(\sigma_1, \dots, \sigma_l) \in$

FIGURE 2. A partitioning for RP_2

$(S_m)^l$ such that σ_1 is the identity permutation, and including in the minimal face associated to $(\sigma_1, \dots, \sigma_l)$ exactly the ranks $im+j$ for $0 \leq i < l, 0 < j < m$ such that $\sigma_i^{-1}(\sigma_{i+1}(j)) > \sigma_i^{-1}(\sigma_{i+1}(j+1))$ (letting $\sigma_0 = \sigma_m$). We omit the justification of this construction, instead showing how to partition the entire complex $\Delta(B_{lm})/S_l \wr S_m$ in a related fashion and verifying the validity of that construction.

The partitioning for $\Delta(B_{lm})/S_l \wr S_m$ will make use a notion of ascents and descents in the facets, based on a labeling for the covering relations. This labeling will give a unique increasing chain on each interval, and the descents will specify which ranks to include in the minimal faces assigned to facets. However, the labeling will not give a lexicographic shelling for three reasons: (1) the labeling is not a chain-labeling, because the label assigned to a covering relation depends not only on the chain below, but also on whether the label is being compared with the one below it or above it in the chain, (2) the increasing chain is not always lexicographically smallest on an interval and (3) we define increasing to mean each pair of consecutive labels is increasing, but because of (1), this is not the same as the entire chain increasing.

We will use a permutation σ that evolves as we proceed upward from $\hat{0}$ to $\hat{1}$ in a saturated chain to play a similar role to the l -tuple $\sigma_1, \dots, \sigma_l$ that appeared immediately after Example 5.1. For each vertex in a saturated chain orbit, σ provides an ordering on the rows from which letters are chosen. Since the choice of permutation σ depends both on the saturated chain orbit and also on the rank within that chain, we will denote the permutation at rank i by $\sigma_i(C)$ when the rank seems necessary to clarify meaning, and we will sometimes omit the rank-indicator.

The permutation $\sigma \in S_m$ is initialized to the identity, and evolves as we proceed from $\hat{0}$ to $\hat{1}$ in a saturated chain by moving a row R in front of all the rows that are currently similar to it (as defined below) whenever a covering relation $T \subset S$ enlarges a set T to S by adding an element from row R .

Before we define row similarity, let us establish a notion of similarity block, though its definition will be inductively intertwined with the definition of row similarity.

Definition 5.2. A series of consecutive covering relations $u_0 \prec \dots \prec u_{st}$ is called a **similarity block** if there is some collection of rows R_1, \dots, R_t that are similar in u_0 and that have each been chosen the same number of times in the saturated chain from $\hat{0}$ to u_0 such that for $0 \leq i < t$ the covering relations $u_{is} \prec \dots \prec u_{(i+1)s}$ all insert copies of the row R_{i+1} .

Notice that the second requirement on u_0 ensures that similarity blocks are non-overlapping, and also note that the rows R_1, \dots, R_t may be listed in any order.

Definition 5.3. *Let us define **similarity** of rows recursively as follows: all of the rows are similar in a saturated chain C at $\hat{0}$. A collection of rows R_1, \dots, R_t which are similar at u will still be similar at v for $u < v$ if every time any one of the rows R_i appears in the interval from u to v , it appears as part of a similarity block involving the rows R_1, \dots, R_t (though this similarity block might continue beyond v or begin prior to u).*

Thus, fewer and fewer rows will be similar to a fixed row R as we proceed from $\hat{0}$ to $\hat{1}$. At the point u when rows R and R' cease to be similar because of R appearing in a similarity block that does not contain R' we have $\sigma_{rk(u)}(R) < \sigma_{rk(u)}(R')$ (and more generally we have $\sigma_j(R) < \sigma_j(R')$ for $j \geq rk(u)$).

Example 5.2. *Consider the saturated chain orbit which sequentially chooses elements from three rows in the following order: 112221132333321. Notice that similarity of rows 1 and 2 lasts until the covering relation inserting the first 3; row 3 ceased to be similar to the other two rows at the covering relation inserting the third 2. Listing those permutations $\sigma_i(C)$ in one-line notation that differ from $\sigma_{i-1}(C)$, we get $\sigma_{init}(C) = 123, \sigma_5 = 213$ and $\sigma_7 = 123$.*

The eventual row order σ_{final} is used to determine descents from wrap-around. At any particular rank, σ reflects the partial evolution from the identity permutation based on row insertion up to this point.

In analogy to our use of $\sigma_i^{-1} \circ \sigma_{i+1}$ (in which we let $\sigma_0 = \sigma_{final}$) following Example 5.1, let us now consider the renormalized permutations $\rho_r(C) = \sigma_{final}^{-1}(C) \circ \sigma_r(C)$. When a covering relation $T \subseteq S$ adds to T an element from a new row, by convention let us choose this element to come from the earliest row not yet chosen. The label for each insertion is the pair $(i, \rho(j))$ where j is the row being chosen and i is the number of times row j has been chosen so far in the chain (including its current selection); the permutation ρ is evaluated either at T (when comparing to a higher covering relation $S \subseteq S'$) or at U (when $T \subseteq S$ is being compared to a lower covering relation $U \subseteq T$).

Definition 5.4. *The **relative transpose order** (cf. [He, p. 25]) on labels $(i, \rho(j))$ is a rule for comparing two consecutive covering relation labels in a saturated chain. We compare covering relations $u < v$ and $v < w$ by comparing their labels $(i_1, \rho_{rk(u)}(j_1))$ and $(i_2, \rho_{rk(u)}(j_2))$, and we say that $(i_1, \rho_{rk(u)}(j_1)) < (i_2, \rho_{rk(u)}(j_2))$ if $i_1 < i_2$ or if $i_1 = i_2$ and $\rho(j_1) < \rho(j_2)$.*

This edge-comparison rule is designed for the sole purpose of specifying which ranks are ascents and which are descents. We call a chain **increasing** on an interval if it has no descents in the relative transpose order on that interval, and likewise a **decreasing chain** must have all descents on the interval. Our partitioning assigns minimal faces G_j to the facets F_j by including in G_j the ranks of the descents in F_j in the relative transpose order.

Theorem 5.1. *This assignment of minimal faces to facets gives a partitioning of $\Delta(B_{lm})/S_l \wr S_m$.*

PROOF. To ensure that our assignment of minimal faces to facets gives a partitioning, we must check (1) that every face belongs to at least one interval $[G_i, F_i]$ and (2) that no face belongs to multiple intervals. To verify (1), we describe in Proposition 5.1 how to extend any face F to a facet F_j whose minimal face G_j is contained in F ; it suffices to show that $\text{supp}(G_j) \subseteq \text{supp}(F)$ since F and G_j are both faces of F_j . The second (much easier) claim is confirmed in Proposition 5.2. \square

Proposition 5.1. *Every face F is contained in an interval $[G_j, F_j]$.*

PROOF. Let us describe how to extend each face F to a facet F_j in such a way that descents in the relative transpose order on labels of F_j only occur at ranks in the support of F . We obtain such an F_j by (1) extending F to a facet \overline{F} in such a way that the extension of each interval of F would be increasing (in the relative transpose order) if $\sigma_{\text{final}}(\overline{F})$ were the identity permutation, then (2) relabeling the rows (since this preserves the facet orbit) so that the relabeling of $\sigma_{\text{final}}(\overline{F})$ written in one-line notation is the identity permutation, then (3) restricting to the resulting representation of the face orbit F (which is no longer in standard form), and finally (4) taking F_j to be the increasing extension of this representation of F , using the fact (to be confirmed in Lemma 5.1) that $\sigma_{\text{final}}(F_j)$ is the identity permutation. Example 5.3 provides an example of this process; notice that $F_j \neq \overline{F}$ in the example, and that the relabeling of \overline{F} has the same set of descents in the relative transpose order as \overline{F} did. Once we check that $\sigma_{\text{final}}(F_j)$ equals the identity permutation, we will know that F_j is increasing on every interval of F , implying $\text{supp}(G_j) \subseteq \text{supp}(F)$. \square

Example 5.3. Let $F = \{1^2, 2\} < \{1^2, 2^3\}$ in $\Delta(B_6)/S_3 \wr S_2$, so then \overline{F} is the saturated chain in which row elements are inserted in the following order: 112221. Notice that $\sigma_{\text{final}}(\overline{F})$ is the adjacent transposition 21. Thus, we relabel by swapping the values 1 and 2, so the relabeled representation of \overline{F} is 221112. This restricts to the new representation for F as $F_{\text{relabel}} = \{2^2, 1\} < \{2^2, 1^3\}$, which extends to F_j by inserting rows as follows: 122112. Notice that $G_j = \{1, 2^2\} < \{1^3, 2^2\} = F$, since F_j has descents at ranks 3, 5, and that F belongs to the interval $[G_j, F_j]$, as desired.

Lemma 5.1. *Each facet F_j constructed in Proposition 5.1 has $\sigma_{\text{final}}(F_j)$ equalling the identity permutation.*

PROOF. We will show that $\sigma_{\text{final}}(F_j)$ has no inversion pairs. Suppose the similarity of rows r and s is broken in \overline{F} on the interval $u < v$ for u, v consecutive elements of the chain F . Let F_{relabel} denote the expression for F in which the rows are permuted so that $\sigma_{\text{final}}(\overline{F})$ is relabeled as the identity permutation. Let us similarly view u and v in this relabeled form. Because this relabeling of \overline{F} sends σ_{final} to a permutation with no inversions, we may conclude that in the relabeled pair $u < v$, that $\sigma_{rk(v)}(r) < \sigma_{rk(v)}(s)$. Since the relabeled \overline{F} is increasing on the relabeled interval $u < v$, we then know that v has more copies of r than of s , and that one of the following properties must hold (letting $v < w$ be the interval of F immediately following $u < v$) to ensure that there is no similarity block for r and s beginning on the interval $u < v$ and concluding on the interval $v < w$:

- (1) the number of new copies of r in $v < w$ is larger than the number of new copies of s on the interval $v < w$
- (2) the interval $u < v$ also inserts letters with larger labels than r, s , implying that these are inserted after the copies of r and s , preventing the continuation of a similarity block to the interval $v < w$
- (3) some row t which has smaller value than r or s (and so would precede any copies of r or s in the interval $v < w$) is inserted in the interval $v < w$, again preventing the continuation of a similarity block to $v < w$

One may easily check that these properties carry over to the intervals $u < v, v < w$ in F_j by virtue of (1) F_j containing the relabeled face F , (2) F_j increasing on intervals, (except possibly from wrap-around) and (3) the fact that similarity of r, s cannot be broken earlier in F_j , by virtue of the same characterization of how similarity is broken applied to the earlier intervals. We conclude that the permutation $\sigma_{final}(F_j)$ has exactly the same inversion pairs as the relabeling of $\sigma_{final}(\bar{F})$, so $\sigma_{final}(F_j)$ is the identity permutation. \square

It is easy to check that each face is included only once in the partitioning.

Proposition 5.2. *There is no overlap among the intervals $[G_j, F_j]$.*

PROOF. If $F \in [G_j, F_j]$, then F_i must be increasing in the relative-transpose order on each interval of F . The only possible flexibility in how to extend F to F_j comes from the choice of presentation of F prior to taking its increasing extension, but at most one such choice will yield σ_{final} which equals the identity permutation, as needed to avoid descents from wrap-around. \square

As a reality check, we computed that the determinant of the incidence matrix for the partitioning of $\text{lk}(\emptyset < \{1, 2\} < \{1^2, 2^2\} < \{1^3, 2^3\})$ is 2 and that the incidence matrix M for the partitioning of $\text{lk}(\emptyset < \{1, 2\} < \{1^2, 2^2\} < \{1^3, 2^3\} < \{1^4, 2^4\})$ has $\det(M) = 8$, consistent with the fact that RP_n only has local 2-torsion.

Question 5.1. *Is the incidence matrix M for this partitioning nonsingular over $\mathbb{Z}/p\mathbb{Z}$ for all $p > 2$? If so, then the partitioning would give a basic set for the subring $k[x_1, \dots, x_{lm}]^{S_l \wr S_m}$ of polynomials that are invariant under the action of $S_l \wr S_m$ for $\text{char}(k) > 2$, by results of [GS] about transferring basic sets. This would imply Cohen-Macaulayness for $\text{char}(k) > 2$.*

We suspect that this question has a negative answer. Notice that M is nonsingular over $\mathbb{Z}/p\mathbb{Z}$ if and only if p does not divide the determinant of M . The incidence matrix M for our partitioning for $\text{lk}(\emptyset < \{1, 2, 3\} < \{1^2, 2^2, 3^2\} < \{1^3, 2^3, 3^3\})$ satisfies $\det(M) = 2^3 \cdot 3^5$, and so is singular over $\mathbb{Z}/3\mathbb{Z}$, suggesting the distinct possibility of local 3-torsion.

6. APPENDIX (BY VIC REINER)

We wish to prove Theorem 2.3. For this purpose, we introduce some notation, which mostly follows that of [GS]:

$R :=$ Stanley-Reisner ring for the Boolean algebra $B_n - \{\emptyset\}$

$= k[y_S : \emptyset \neq S \subseteq [n]] / I$

where I is the ideal generated by all products $y_S y_T$

with S, T incomparable subsets of $[n]$

$R' := k[x_1, \dots, x_n]$

$G =$ a subgroup of the symmetric group S_n ,

acting on both R, R' by permuting subscripts.

$R^G, (R')^G$ the corresponding invariant subrings.

$T :=$ the transfer map $R \rightarrow R'$ from [GS], mapping $y_S \mapsto \prod_{i \in S} x_i$,

then extending multiplicatively to non-vanishing monomials

in R' , then further extending k -linearly to all of R' .

$\theta_i := \sum_{S: |S|=i} y_S \in R$

$k[\theta] := k[\theta_1, \dots, \theta_n] \subset R$

$e_i :=$ the i^{th} elementary symmetric function in x_1, \dots, x_n

$= T(\theta_i)$

$k[e] := k[e_1, \dots, e_n] \subset R'$

$= T(k[\theta]).$

Theorem 6.1. *If R^G is a Cohen-Macaulay ring, then $(R')^G$ is also a Cohen-Macaulay ring.*

PROOF. If R^G is Cohen-Macaulay, then the h.s.o.p. $\theta_1, \dots, \theta_n$ is a regular sequence, so R^G is a free module over the polynomial ring $k[\theta] := k[\theta_1, \dots, \theta_n]$. Furthermore, we can choose a basis for this free module consisting of elements η_1, \dots, η_t which are homogeneous with respect to the fine \mathbb{N}^n -grading on R^G (choosing any η'_i s which are finely homogeneous liftings of a k -vector space basis for $R^G / (k[\theta]_+)$ will work).

We wish to show that $T(\eta_1), \dots, T(\eta_t)$ comprise a $k[e]$ -basis for $(R')^G$ as a free $k[e]$ -module, which would then show that $(R')^G$ is Cohen-Macaulay. We first argue by a comparison of Hilbert series that one only needs to show that $T(\eta_1), \dots, T(\eta_t)$ span. Since T is a G -equivariant k -vector space isomorphism (but not a ring isomorphism!) from R to R' , it restricts to a k -vector space isomorphism from R^G to $(R')^G$. If, for the moment, we coarsely \mathbb{N} -grade R^G by applying the usual specialization to its fine \mathbb{N}^n -grading (i.e. so that y_S has degree $|S|$), then T also respects the polynomial gradings on each side. This implies R^G and $(R')^G$ have the same Hilbert series. Hence the fact that η_1, \dots, η_t form a free $k[\theta]$ -basis for R^G implies that the degrees of $T(\eta_1), \dots, T(\eta_t)$ are such that there are the right number of $k[e]$ -linear combinations of them in each degree to form a basis of $(R')^G$. If we can show that $T(\eta_1), \dots, T(\eta_t)$ do span $(R')^G$ as a $k[e]$ -module, we would then know

that these $k[e]$ -linear combinations give a k -basis in each degree, so they would form a $k[e]$ -basis for $(R')^G$.

For the spanning argument, since $(R')^G$ is spanned as a k -vector space by G -orbit sums $G(x^\alpha)$ of monomials $x^\alpha \in R'$, we only need to show that such elements are in the $k[e]$ -span of the $T(\eta_i)'s$. Let $G(x^\alpha)$ be any such G -orbit sum. Let $T^{-1}(G(x^\alpha))$ have an expression in R^G as follows:

$$T^{-1}(G(x^\alpha)) = \sum_i \eta_i p_i(\theta) \quad (*)$$

for some polynomials p_i in the θ' s.

We will show that

$$G(x^\alpha) - \sum_i T(\eta_i) p_i(e) \quad (**)$$

is a sum of monomials x^β whose “shapes” (as defined in [GS, p.178]) are all lower in the dominance order than the shape of x^α , using [GS, Lemma 9.1], and then be done by induction on the dominance order.

To see this, note that the shape of x^α (and every other monomial occurring in $G(x^\alpha)$) is the same as the fine grading of the element $T^{-1}(G(x^\alpha))$, so that in expression (*), we may assume that every term in the sum has this same \mathbb{N}^n -grading (by \mathbb{N}^n -gradedness of R^G). Then [GS, Lemma 9.1] tells us that every monomial one obtains by multiplying out the terms in the sum in (**) will have shape less than or equal to that of x^α in dominance order, and that those whose shapes match those of x^α exactly correspond to the terms in (*), so they all cancel with terms in $G(x^\alpha)$ due to the equality (*). The shapes of the remaining non-cancelling monomials in (**) are all strictly lower in dominance order. \square

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